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The issue of black hole entropy is reexamined within a finite lattice framework along the lines of Wheeler, 't Hooft and Susskind, with an additional criterion to identify physical horizon states contributing to the entropy. As a consequence, the degeneracy of physical states is lower than that attributed normally to black holes. This results in corrections to the Bekenstein-Hawking area law that are logarithmic in the horizon area. Implications for the holographic entropy bound on bounded spaces are discussed. Theoretical underpinnings of the criterion imposed on the states, based on the ‘quantum geometry’ formulation of quantum gravity, are briefly explained.

I. INTRODUCTION

The notion of black hole entropy, introduced by Bekenstein [1] on the basis of the laws of black hole mechanics [2] and gleaned insights from (classical) information theory, is one of the most profound in black hole physics. The issue of which microstates contribute to the entropy has remained a challenging one, since these states should ostensibly appear in a quantum theory that includes gravitation. In the absence (even now) of a complete quantum theory of gravitation, a measure of the entropy was suggested on semiclassical grounds [1], [3] as being equal to a quarter of the horizon area of the black hole. Deeply engaging insights into this unexpected dependence of entropy on surface area (rather than volume) have appeared early in the last decade. The ‘It from bit’ picture of Wheeler [4] contains the germ of these insights which have subsequently been substantially refined [5]–[6], leading to the bold proposal of the principle of Holography, as applied to quantum gravity. In this paper, some of these insights are reexamined from a somewhat different standpoint. The departure from standard lore appears to bring into the fold the first truly ‘quantum gravity’ aspects beyond the semiclassical Bekenstein-Hawking Area Law (BHAL).

The paper is organized as follows: in Section 2, we survey the basic ingredients of the ‘It from bit’ picture; in Section 3, we discuss the criterion we impose on the space of states to identify the physical Hilbert space of horizon states; the dimensionality of this physical Hilbert space yields the entropy of the black hole which has a $\log(\text{area})$ correction over and above the BHAL. This is argued to lead to an upper bound on the entropy of all bounded three-spaces (where the boundary is S^2) via the holographic principle. In Section 4, we *derive* our physical subspace criterion of the earlier section on the basis of the microscopic theory of Quantum Geometry. Our concluding remarks are presented in Section 5.

II. ‘IT FROM BIT’

Consider a two dimensional finite ‘floating lattice’ with plaquettes approximately the size of a Planck area ($\sim l_P^2$) covering the spherical horizon of an eternal non-rotating four dimensional black hole. The black hole is assumed to be macroscopic¹ in that the classical area of its horizon $A_S/l_P^2 \gg 1$. Assume that binary variables (‘bits’, ‘Boolean variables’ or ‘pixels’) are distributed randomly on this lattice. Typically, these could be elementary spin 1/2 variables or doublets of an $SU(2)$ group. Assume also that the size of the lattice is characterized by a finite large even integer p . Clearly, the Hilbert space of quantum states defined by these spin 1/2 variables has a dimensionality $\mathcal{N}(p) = 2^p$. It follows that the number of degrees of freedom characterizing the horizon is given by $N \equiv \log \mathcal{N}(p) = p \log 2$. Given the relation between entropy and number of degrees of freedom, the former is also proportional to the size p of the lattice. By assumption, $p \gg 1$; in the limit of very large p , the lattice can be taken to approximate the macroscopic horizon of the black hole. One would then expect that the classical horizon area A_S would satisfy $A_S/l_P^2 = \xi p$ where $\xi = O(1)$. For a choice $\xi = 4 \log 2$, one obtains for the entropy $S_{bh} \equiv N = A_S/4l_P^2$ which is the famous BHAL.

The generality of the above scenario makes it appealing vis-a-vis a quantum theory of black holes in particular and of quantum gravity in general. There is however one crucial aspect of any quantum approach to black hole physics which seems to have been missed in the above, – the aspect of symmetry. Indeed, the mere random distribution of spin 1/2 (binary) variables on the lattice which approximates the black hole horizon, without regard to possible

¹Planck-size or primordial black holes fall outside the purview of this work

symmetries, possibly leads to a far bigger space of states than the *physical* Hilbert space, and hence to an overcounting of the number of the degrees of freedom, i.e., a larger entropy.

III. THE PHYSICAL HILBERT (SUB)SPACE

A. A natural symmetry criterion

But what is the most plausible symmetry that one can impose on states so as to identify the physical subspace? Recall that the elementary variables are binary or spin 1/2 variables which can be considered to be in the fundamental doublet representation of an $SU(2)$ Lie Algebra. On very general grounds then, the most natural symmetry of the physical subspace must be this $SU(2)$. One is thus led to a symmetry criterion which identifies the physical Hilbert space \mathcal{H}_S of horizon states contributing to black hole entropy: *\mathcal{H}_S consists of states, composed of elementary $SU(2)$ doublets, which are $SU(2)$ singlets.* Observe that this criterion has no allusions whatsoever to any specific proposal for a quantum theory of gravitation. Nor does it involve any gauge redundancies (or any other infinite dimensional symmetry like conformal invariance) at this point. It is the most natural choice for the symmetry of physical horizon states simply because in the ‘It from bit’ picture, the basic variables are spin 1/2 variables. Were they of higher multiplicity than 2, the Lie Algebra of symmetries might have been likewise different. Later on we shall show however that this symmetry arises very naturally in the Quantum Geometry approach to black hole physics. It will emerge from that approach that horizon states of large macroscopic black holes are best described in terms of spin 1/2 variables at the punctures of a punctured two-sphere which represents (a spatial slice of) the event horizon.

B. Dimensionality of \mathcal{H}_S

The criterion of $SU(2)$ invariance leads to a simple way of counting the dimensionality of the physical Hilbert space, as has already been shown [7]. For p variables, this number is given by

$$\dim \mathcal{H}_S \equiv \mathcal{N}(p) = \binom{p}{p/2} - \binom{p}{(p/2-1)} \quad (1)$$

There is a simple intuitive way to understand the result embodied in (1). This formula counts the number of ways of making $SU(2)$ singlets from p spin 1/2 representations. The first term corresponds to the number of states with net J_3 quantum number $m = 0$ constructed by placing $m = \pm 1/2$ on the punctures. However, this term by itself *overcounts* the number of $SU(2)$ singlet states, because even non-singlet states (with net integral spin, for p is an even integer) have a net $m = 0$ sector. Beside having a sector with total $m = 0$, states with net integer spin have, of course, a sector with overall $m = \pm 1$ as well. The second term basically eliminates these non-singlet states with $m = 0$, by counting the number of states with net $m = \pm 1$ constructed from $m = \pm 1/2$ on the p sites. The difference then is the net number of $SU(2)$ singlet states that represents the dimensionality of \mathcal{H}_S .

It may be pointed out that the first term in (1) also has another interpretation. It counts the number of ways binary variables corresponding to spin-up and spin-down can be placed on the sites to yield a vanishing total spin. Alternatively, one can think of the binary variables as unit positive and negative $U(1)$ charges; the first term in (1) then corresponds to the dimensionality of the Hilbert space of $U(1)$ *invariant* states. As already shown in [7], this corresponds to a *binomial* rather than a random distribution of binary variables.

C. Large p approximation and black hole entropy

In the limit of very large p , one can evaluate the factorials in (1) using the Stirling approximation. One obtains

$$\mathcal{N}(p) \approx \frac{2^p}{p^{\frac{3}{2}}} \quad (2)$$

Clearly, the dimensionality of the physical Hilbert space is smaller than what one had earlier, as would be an obvious consequence of imposing $SU(2)$ symmetry. Using the relation between p and the classical horizon area A_S discussed in the last section, with the constant ξ chosen to take the same value as in that section, (2) can be shown [7] to lead to the following formula for black hole entropy,

$$S_{bh} \equiv \log \mathcal{N}(p) \approx \frac{A_S}{4l_P^2} - \frac{3}{2} \log \left(\frac{A_S}{4l_P^2} \right) + \text{const.} + O(A_S^{-1}). \quad (3)$$

The general nature of the assumptions underlying the derivation of eq. (3) can hardly be overemphasized. Nowhere has any particular aspect of a *microscopic* theory of quantum gravity been used. Nor did we need to appeal to any specific infinite dimensional symmetry of the classical horizon geometry, and formulas pertaining to such an invariance. In spite of this, the physical subspace criterion discussed earlier in this section, leads unequivocally to the BHAL, and a correction term logarithmic in the horizon area, with the coefficient $-3/2$. Of course, the constant ξ was chosen to have a fixed value, in order to fix the normalization of the BHAL. Observe though that the coefficient of the $\log(\text{area})$ correction is unaffected by the choice of ξ . There are further sub-leading corrections that are constant and inversely proportional to powers of horizon area, as an expansion for large areas should entail.

This result is of course not new; it has been derived earlier [8] within the context of quantum geometry, where the correct normalization of the BHAL dictates a choice of the Barbero-Immirzi parameter. Once again, the coefficient of the logarithmic correction is independent of this choice. Rather compelling arguments have been presented (from a perspective different from ours) [9] that the correction could be of a ‘universal’ character. Our discussion above would lend credence to such an inference on more general grounds. But, perhaps of similar significance is the feature that *kinematically*, quantum horizon states of large macroscopic non-rotating black holes appear to have a rather simple description in terms of $SU(2)$ singlet states of a (two dimensional) lattice of spin $1/2$ variables. It turns out that such a description is not restricted to static black hole horizon states, but in fact, describe Isolated Horizons [10], [11] of macroscopic size as well.

D. Holography and the entropy bound

Having identified the kinematical quantum states characterising a black hole horizon, the question that immediately comes to mind is whether there are other states that describe black hole physics. Although the ‘It from bit’ picture tends to imply that the entire information lies with the horizon states, this has been more sharply articulated in the so-called Holographic Principle [5]. According to this principle, the horizon states exhaust the Hilbert space of a black hole, encoding the entire information of gravitationally collapsed matter in terms of macroscopic observables like the horizon area. The entropy of a black hole can then be taken to represent the *maximal* possible entropy of a spacetime whose spatial slice has a boundary that coincides with the intersection of this spatial slice with the horizon. Now, it can be shown [7] that eq. (3) actually translates into a bound on black hole entropy, given by

$$S_{max} = \ln \left(\frac{\exp S_{BH}}{S_{BH}^{3/2}} \right) \quad (4)$$

Thus, it follows that *all* 3-spaces with boundary have an entropy bounded from above by (4). That this is extremely plausible follows from the following argument, based on *reductio ad absurdum* [12]: we assume, for simplicity that the spatial slice of the boundary of an asymptotically flat spacetime has the topology of a 2-sphere on which is induced a spherically symmetric 2-metric. Let this spacetime contain an object whose entropy exceeds the entropy bound given in eq. (4). Certainly, such a spacetime cannot have a black hole horizon as a boundary, since then, its entropy would have been subject to (4). But, in that case, its energy should be less than that of a black hole which has the 2-sphere as its horizon. Let us now add energy to the system, so that it does transform adiabatically into a black hole with the said horizon, but without affecting the entropy of the exterior. But we have already seen above that a black hole with such a horizon must obey the bound (4); it follows that the starting assumption of the system having an entropy exceeding (4) must be incorrect. Thus, we have indeed obtained an upper bound on the entropy of a large class of spacetimes. Notice that this bound *tightens* the semiclassical Bekenstein bound [13], which is of course expected because of its quantum kinematical underpinning. We now turn to an exposé of this underlying structure within the framework of Quantum Geometry.

IV. ENTROPY FROM QUANTUM GEOMETRY

The presentation in this section closely follows ref. [7]; we consider generic 3+1 dimensional Isolated Horizons without rotation, on which one assumes an appropriate class of boundary conditions [10]. These boundary conditions require that the gravitational action be augmented by the action of a Chern-Simons theory living on the isolated horizon. Boundary states of the Chern-Simons theory contribute to the entropy. These states correspond to conformal

blocks of the two-dimensional Wess-Zumino model that lives on the spatial slice of the horizon, which is a 2-sphere of area A_H . The dimensionality of the boundary Hilbert space has been calculated thus [8], [14] by counting the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model, for arbitrary level k and number of punctures p on the 2-sphere. We shall show, from the formula for the number of conformal blocks specialized to macroscopic black holes characterized by large k and p [8], that eq. (3) ensues.

Let us start with the formula for the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model that lives on the punctured 2-sphere. For a set of punctures \mathcal{P} with spins $\{j_1, j_2, \dots, j_p\}$ at punctures $\{1, 2, \dots, p\}$, this number is given by [14]

$$N^{\mathcal{P}} = \frac{2}{k+2} \sum_{r=0}^{k/2} \frac{\prod_{l=1}^p \sin\left(\frac{(2j_l+1)(2r+1)\pi}{k+2}\right)}{\left[\sin\left(\frac{(2r+1)\pi}{k+2}\right)\right]^{p-2}}. \quad (5)$$

Observe now that Eq. (5) can be rewritten as a multiple sum,

$$N^{\mathcal{P}} = \left(\frac{2}{k+2}\right) \sum_{l=1}^{k+1} \sin^2 \theta_l \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_p=-j_p}^{j_p} \exp\{2i(\sum_{n=1}^p m_n) \theta_l\}, \quad (6)$$

where, $\theta_l \equiv \pi l/(k+2)$. Expanding the $\sin^2 \theta_l$ and interchanging the order of the summations, this becomes

$$N^{\mathcal{P}} = \sum_{m_1=-j_1}^{j_1} \cdots \sum_{m_p=-j_p}^{j_p} \left[\bar{\delta}_{(\sum_{n=1}^p m_n),0} - \frac{1}{2} \bar{\delta}_{(\sum_{n=1}^p m_n),1} - \frac{1}{2} \bar{\delta}_{(\sum_{n=1}^p m_n),-1} \right], \quad (7)$$

where, we have used the standard resolution of the periodic Kronecker deltas in terms of exponentials with period $k+2$,

$$\bar{\delta}_{(\sum_{n=1}^p m_n),m} = \left(\frac{1}{k+2}\right) \sum_{l=0}^{k+1} \exp\{2i[(\sum_{n=1}^p m_n) - m]\theta_l\}. \quad (8)$$

Our interest focuses on the limit of large k and p , appropriate to macroscopic black holes of large area. Observe, first of all, that as $k \rightarrow \infty$, the periodic Kronecker delta's in (8) reduce to ordinary Kronecker deltas,

$$\lim_{k \rightarrow \infty} \bar{\delta}_{m_1+m_2+\dots+m_p,m} = \delta_{m_1+m_2+\dots+m_p,m}. \quad (9)$$

In this limit, the quantity $N^{\mathcal{P}}$ counts the number of $SU(2)$ singlet states, rather than $SU(2)_k$ singlets states. For a given set of punctures with $SU(2)$ representations on them, this number is larger than the corresponding number for the affine extension. It is here that one makes contact with the ‘physical subspace criterion’ introduced in Section 3: the $SU(2)$ invariance is completely natural within this microscopic approach.

Next, recall that the eigenvalues of the area operator for the horizon, lying within one Planck area of the classical horizon area A_H , are given by

$$\hat{A}_H \Psi_S = 8\pi\beta l_P^2 \sum_{l=1}^p [j_l(j_l+1)]^{\frac{1}{2}} \Psi_S, \quad (10)$$

where, l_P is the Planck length, j_l is the spin on the l th puncture on the 2-sphere and β is the Barbero-Immirzi parameter [15]. We consider a large fixed classical area of the horizon, and ask what the largest value of number of punctures p should be, so as to be consistent with (10); this is clearly obtained when the spin at *each* puncture assumes its lowest nontrivial value of $1/2$, so that, the relevant number of punctures p is given by

$$p = \frac{A_H}{4l_P^2} \frac{\beta_0}{\beta}, \quad (11)$$

where, $\beta_0 = 1/\pi\sqrt{3}$. We are of course interested in the case of very large p appropriate to a macroscopic black hole. Observe that

Now, with the spins at all punctures set to $1/2$, the number of states for this set of punctures \mathcal{P} is given by

$$N^{\mathcal{P}} = \sum_{m_1=-1/2}^{1/2} \cdots \sum_{m_p=-1/2}^{1/2} \left[\delta_{(\sum_{n=1}^p m_n),0} - \frac{1}{2} \delta_{(\sum_{n=1}^p m_n),1} - \frac{1}{2} \delta_{(\sum_{n=1}^p m_n),-1} \right] \quad (12)$$

The summations can now be easily performed, with the result given *precisely* by the *rhs* of eq. (3).

This establishes on a microscopic basis the validity of the extension of the ‘It from bit’ picture proposed by us in the last section. The central role played by variables in the doublet representation of a (global) $SU(2)$ group, which we identified with the binary variables on the lattice approximating the horizon, is now clarified. This completes the derivation of our physical space criterion and the ensuing entropy formula and holographic bound on the basis of a quantum kinematical formulation.

We close this section with a remark on the Barbero-Immirzi parameter which has often led to confusing statements regarding a so-called ‘ambiguity’ within the quantum geometry approach. The Barbero-Immirzi parameter is an intrinsic aspect of a Hamiltonian formulation of general relativity based on connection variables [16]. Its appearance in the quantum theory is akin to the appearance of the θ parameter in quantum Yang-Mills theory in that it signifies the existence of a one-parameter family of quantum theories corresponding to the same classical theory. The actual value of θ can only be decided by experiment. Here in the case of black hole kinematics, without recourse to experiment, we have followed [10] to fix it to match the coefficient of the BHAL which we believe as the correct lowest order result for macroscopic black holes. This ‘fix’ works for all non-rotating, generic, four-dimensional black holes. Whether it works for rotating black holes as well, remains to be seen. But until a contradiction becomes manifest, there is no reason to suspect that the quantum kinematics described in this approach has any inherent deficiency.

V. CONCLUSIONS

Even though the physical subspace criterion becomes automatic in the quantum geometry approach, because of special properties of two dimensional WZW theories at large level and large number of punctures, we would like to emphasize that it may not be restricted to one particular microscopic construct. Our proposal of this criterion in Section 3 was made on quite general grounds, and we have shown that the leading quantum (kinematical) correction follows uniquely from this criterion, without requiring an underlying microscopic substructure. The existence of this latter substructure, of course, places the criterion on far firmer grounds than would have been otherwise possible. However, insofar as macroscopic black holes are concerned, the full machinery of the quantum geometry approach does not appear to be exigent. It is another question as to how our criterion by itself can be used to analyse situations which are qualitatively different in that quantum *dynamical* effects become more important. Our guess is that such situations require a fuller quantum general relativity than what we have at our disposal at the moment.

We should also mention that some recent authors [17] have alleged that the universality proofs given in [9] of the coefficient of the $\log(\text{area})$ correction are in fact not technically complete. We have no comments to offer on this from our heuristic perspective.

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